Elementary Mathematics

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Unit III

Trigonometry

Power of Sines and Cosines of θ in terms of functions of multiples of θ.
Expansions of $\sin\theta$ and $\cos\theta$ in a series of ascending powers of θ.

Expansion of $\cos^n\theta$ and $\sin^n\theta$.

Expansion of $\cos^n\theta$, $\sin^n\theta$ and $\cos^m\theta \cdot \sin^n\theta$ in terms of Cosines or sines of multiple angles of θ as noted given below:

$\cos^n\theta$	Always in terms of Cosines	
$\sin^n\theta$	In terms of Cosines,	if n is a even number
	In terms of sines,	if n is an odd number
$\cos^m\theta \cdot \sin^n\theta$	In terms of Cosines	if n is ~~an~~ a even number
	In terms of Sines	if n is an odd number

~~Let~~ Demoivre's Theorem

Let $x = \cos\theta + i\sin\theta$, then

$$x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Note: (1) $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

(2) $x + \frac{1}{x} = 2\cos\theta$

(3) $x - \frac{1}{x} = 2i\sin\theta$

(4) $x^n + \frac{1}{x^n} = 2\cos n\theta$

(5) $x^n - \frac{1}{x^n} = 2i\sin n\theta$

Points to Remember

① **Binomial expansion**

If n is a positive integer, then

① $(x+a)^n = x^n + \binom{n}{1} x^{n-1} \cdot a + \binom{n}{2} x^{n-2} a^2 + \dots + a^n$.

(ii) $(x-a)^n = x^n - \binom{n}{1} x^{n-1} a + \binom{n}{2} x^{n-2} a^2 - \dots + (-1)^n a^n$

② $\left(x+\frac{1}{x}\right)^n = x^n + \binom{n}{1} x^{n-1}\left(\frac{1}{x}\right) + \binom{n}{2} x^{n-2}\left(\frac{1}{x^2}\right) + \dots + \left(\frac{1}{x^n}\right)$

③ $\left(x-\frac{1}{x}\right)^n = x^n - \binom{n}{1} x^{n-1}\left(\frac{1}{x}\right) + \binom{n}{2} x^{n-2}\left(\frac{1}{x^2}\right) + \dots + (-1)^n\left(\frac{1}{x^n}\right)$

④ Number of terms involved in the expansion of $(x+a)^n$ is $(n+1)$ terms.

⑤ $(r+1)^{th}$ term in the expansion of $(x+a)^n$ is

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r \; ; \quad r = 0, 1, 2 \dots n$$

⑥ In the expansion of ~~$(a+b)^n$~~ $(x+a)^n$; $n \in N$

(i) If n is even, the middle term is

$$T_{\frac{n}{2}+1} = \binom{n}{n/2} x^{n-\frac{n}{2}} a^{\frac{n}{2}}$$

Note: $T_{\frac{n}{2}+1} = T_{\frac{n+2}{2}}$

(ii) If n is odd, the two middle terms are

$$T_{\frac{n-1}{2}+1} \text{ and } T_{\frac{n+1}{2}+1}$$

$T_{\frac{n-1}{2}+1} = T_{\frac{n+1}{2}}$

$T_{\frac{n+1}{2}+1} = T_{\frac{n+3}{2}}$

⑦ Coefficients of the terms of $(x+a)^n$ are equidistant from beginning and end are equal.

ie, $\binom{n}{r} = \binom{n}{n-r}$

$\binom{n}{n} = \binom{n}{0}$

$\binom{n}{n-1} = \binom{n}{1}$ and so on.

Note:
$\binom{n}{0} = \binom{n}{n} = 1$
$\binom{n}{1} = n$
$\binom{n}{r} = \binom{n}{n-r}$

Expansion of $\cos^n\theta$

W.K.T

$$2\cos\theta = x+\frac{1}{x}$$

Note

$$x+\frac{1}{x} = 2\cos\theta$$
$$x^2+\frac{1}{x^2} = 2\cos 2\theta$$
$$x^3+\frac{1}{x^3} = 2\cos 3\theta$$
$$\vdots$$

Raising to the Power 'n'

$$2^n\cdot\cos^n\theta = \left(x+\frac{1}{x}\right)^n$$

$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}\left(\frac{1}{x}\right) + \binom{n}{2}x^{n-2}\left(\frac{1}{x^2}\right) + \binom{n}{3}x^{n-3}\left(\frac{1}{x^3}\right) + \cdots$$

$$+\binom{n}{n-1}x\left(\frac{1}{x^{n-1}}\right) + \binom{n}{n}\frac{1}{x^n}$$

Since, the coefficients of the terms are equidistant from beginning and end are equal

$$\therefore 2^n\cos^n\theta = \binom{n}{0}\left(x^n+\frac{1}{x^n}\right) + \binom{n}{1}\left(x^{n-2}+\frac{1}{x^{n-2}}\right) + \binom{n}{2}\left(x^{n-4}+\frac{1}{x^{n-4}}\right)$$

$$+\cdots+\binom{n}{n/2}$$

(i) If n is even

$$\cos^n\theta = \frac{1}{2^n}\left\{\binom{n}{0}\left(x^n+\frac{1}{x^n}\right) + \binom{n}{1}\left(x^{n-2}+\frac{1}{x^{n-2}}\right) + \cdots + \binom{n}{n/2}\right\}$$

$$= \frac{1}{2^n}\left\{2\cos n\theta + \binom{n}{1}2\cos(n-2)\theta + \binom{n}{2}2\cos(n-4)\theta + \cdots + \binom{n}{n/2}\right\}$$

(ii) If n is odd

The Number of terms in the expansion is $(n+1)$ is even

There are two middle terms $T_{\frac{n+1}{2}}$ and $T_{\frac{n+3}{2}}$

$$\cos^n\theta = \frac{1}{2^n}\left\{\binom{n}{0}\left(x^n+\frac{1}{x^n}\right) + \binom{n}{1}\left(x^{n-2}+\frac{1}{x^{n-2}}\right) + \binom{n}{2}\left(x^{n-4}+\frac{1}{x^{n-4}}\right)\right.$$

$$\left.+\cdots+\binom{n}{\frac{n-1}{2}}\left(x+\frac{1}{x}\right)\right.$$

$$\cos^n\theta = \frac{1}{2^n}\left\{2\cos n\theta + \binom{n}{1}2\cos(n-2)\theta + \binom{n}{2}2\cos(n-4)\theta\right.$$

$$\left.+\cdots+\binom{n}{\frac{n-1}{2}}2\cos\theta\right\}$$

Pascal's Rule to find the binomial Coefficients:

For the Expansion	Index										
$(x+a)^0$	0	1									
$(x+a)^1$	1	1	1								
$(x+a)^2$	2	1	2	1							
$(x+a)^3$	3	1	3	3	1						
$(x+a)^4$	4	1	4	6	4	1					
$(x+a)^5$	5	1	5	10	10	5	1				
$(x+a)^6$	6	1	6	15	20	15	6	1			
$(x+a)^7$	7	1	7	21	35	35	21	7	1		
$(x+a)^8$	8	1	8	28	56	70	56	28	8	1	
$(x+a)^9$	9	1	9	36	84	126	126	84	36	9	1

Example 3.1 Express $\cos^6\theta$ in terms of Cosines of multiples of θ

Solution Let $x = \cos\theta + i\sin\theta$ then $\frac{1}{x} = \cos\theta - i\sin\theta$

So that $x + \frac{1}{x} = 2\cos\theta$ and $x^n + \frac{1}{x^n} = 2\cos n\theta$

$$\left(x+\frac{1}{x}\right)^6 = (2\cos\theta)^6$$

$$\Rightarrow 2^6\cos^6\theta = \left(x+\frac{1}{x}\right)^6$$

By Pascal triangle

$$= 1\cdot x^6 + 6x^5\left(\frac{1}{x}\right) + 15x^4\left(\frac{1}{x^2}\right) + 20x^3\left(\frac{1}{x^3}\right) + 15x^2\left(\frac{1}{x^4}\right) + 6x\left(\frac{1}{x^5}\right) + 1\cdot\frac{1}{x^6}$$

$$= x^6 + 6x^4 + 15x^2 + 20 + 15\left(\frac{1}{x^2}\right) + 6\left(\frac{1}{x^4}\right) + \frac{1}{x^6}$$

Combining the coefficients of the terms, ~~are~~ equidistance from the beginning and end

$$= 1\left(x^6+\frac{1}{x^6}\right)+6\left(x^4+\frac{1}{x^4}\right)+15\left(x^2+\frac{1}{x^2}\right)+20$$

$$= 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$$

$$2^6 \cos^6\theta = 2\{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\}$$

$$\boxed{\cos^6\theta = \frac{1}{2^5}\{\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\}}$$

$$\boxed{\frac{2}{2^6} = \frac{1}{2^5}}$$

Example 3.2 Prove that $\cos^8\theta = \frac{1}{2^7}\{\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35\}$

Proof:

Let $x = \cos\theta + i\sin\theta$ then $\frac{1}{x} = \cos\theta - i\sin\theta$

So that $x + \frac{1}{x} = 2\cos\theta \rightarrow (1)$

and $x^n + \frac{1}{x^n} = 2\cos n\theta$ where, n is a Positive ~~integer~~ integer

$$x^8 + \frac{1}{x^8} = 2\cos 8\theta$$

$$x^6 + \frac{1}{x^6} = 2\cos 6\theta$$

$$x^4 + \frac{1}{x^4} = 2\cos 4\theta$$

$$x^2 + \frac{1}{x^2} = 2\cos 2\theta$$

$2\cos\theta = x + \frac{1}{x}$ $\{\because$ from (1)

Raising to the power 8

$$2^8\cos^8\theta = \left(x+\frac{1}{x}\right)^8$$

Here index $n=8$

$$= 1x^8 + 8x^7\left(\frac{1}{x}\right) + 28x^6\left(\frac{1}{x^2}\right) + 56x^5\left(\frac{1}{x^3}\right) + 70x^4\left(\frac{1}{x^4}\right) + 56x^3\left(\frac{1}{x^5}\right) + 28x^2\left(\frac{1}{x^6}\right) + 8x\left(\frac{1}{x^7}\right) + \frac{1}{x^8}$$

$$= x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + 56\left(\frac{1}{x^2}\right) + 28\left(\frac{1}{x^4}\right) + 8\left(\frac{1}{x^6}\right) + \frac{1}{x^8}$$

$$= \left(x^8 + \frac{1}{x^8}\right) + 8\left(x^6 + \frac{1}{x^6}\right) + 28\left(x^4 + \frac{1}{x^4}\right) + 56\left(x^2 + \frac{1}{x^2}\right) + 70$$

$$= 2\cos 8\theta + 8(2\cos 6\theta) + 28(2\cos 4\theta) + 56(2\cos 2\theta) + 70$$

$$2^8 \cos^8\theta = 2\{\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35\}$$

$$\boxed{\cos^8\theta = \frac{1}{2^7}\{\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35\}}$$

Example 3.3

Prove that $\cos^7\theta = \frac{1}{2^6}\{\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta\}$

Proof

Let $x = \cos\theta + i\sin\theta$ then $\frac{1}{x} = \cos\theta - i\sin\theta$

So that $x + \frac{1}{x} = 2\cos\theta$ and $x^n + \frac{1}{x^n} = 2\cos n\theta$

$$2\cos\theta = x + \frac{1}{x}$$

Raising to the Power 7

$$(2\cos\theta)^7 = \left(x + \frac{1}{x}\right)^7$$

$$= x^7 + 7x^6\left(\frac{1}{x}\right) + 21x^5\left(\frac{1}{x^2}\right) + 35x^4\left(\frac{1}{x^3}\right) + 35x^3\left(\frac{1}{x^4}\right) + 21x^2\left(\frac{1}{x^5}\right) + 7x\left(\frac{1}{x^6}\right) + 1\left(\frac{1}{x^7}\right)$$

$$= x^7 + 7x^5 + 21x^3 + 35x + 35\left(\frac{1}{x}\right) + 21\left(\frac{1}{x^3}\right) + 7\left(\frac{1}{x^5}\right) + \left(\frac{1}{x^7}\right)$$

$$= \left(x^7 + \frac{1}{x^7}\right) + 7\left(x^5 + \frac{1}{x^5}\right) + 21\left(x^3 + \frac{1}{x^3}\right) + 35\left(x + \frac{1}{x}\right)$$

$$= 2\cos 7\theta + 7(2\cos 5\theta) + 21(2\cos 3\theta) + 35(2\cos\theta)$$

$$2^7\cos^7\theta = 2\{\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta\}$$

$$\boxed{\cos^7\theta = \frac{1}{2^6}\{\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta\}}$$

Example 3.4 Expand $\sin^5\theta$

Solution Let $x = \cos\theta + i\sin\theta$ then $\frac{1}{x} = \cos\theta - i\sin\theta$

So that, $x - \frac{1}{x} = 2i\sin\theta$

and $x^n - \frac{1}{x^n} = 2i\sin n\theta$

$$(2i\sin\theta)^5 = \left(x - \frac{1}{x}\right)^5$$

From the Pascal triangle Binomial Coefficients for the index 5 are 1, 5, 10, 10, 5, 1

$$= 1\cdot x^5 - 5x^4\left(\frac{1}{x}\right) + 10x^3\left(\frac{1}{x^2}\right) - 10x^2\left(\frac{1}{x^3}\right) + 5x\left(\frac{1}{x^4}\right) - 1\left(\frac{1}{x^5}\right)$$

$$= x^5 - 5x^3 + 10x - 10\left(\frac{1}{x}\right) + 5\left(\frac{1}{x^3}\right) - \frac{1}{x^5}$$

Combining the like coefficients, equidistant from first and last

$$= \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$

$$2^5 i^5 \sin^5\theta = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$$

$$32 i \sin^5\theta = 2i\{\sin 5\theta - 5\sin 3\theta + 10\sin\theta\}$$

$$\boxed{\sin^5\theta = \frac{1}{16}\{\sin 5\theta - 5\sin 3\theta + 10\sin\theta\}}$$

$i^5 = i\cdot i^4$

$= i\cdot 1$

$i^5 = i$

Example 3.5 Express $\sin^6\theta$ in a terms of Cosine.

Solution Let $x = \cos\theta + i\sin\theta$ then $\frac{1}{x} = \cos\theta - i\sin\theta$

So that, $x + \frac{1}{x} = 2\cos\theta$ and $x - \frac{1}{x} = 2i\sin\theta$

$$x^n + \frac{1}{x^n} = 2\cos n\theta \qquad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

we have,

$$2i\sin\theta = x - \frac{1}{x}$$

$$(2i\sin\theta)^6 = \left(x - \frac{1}{x}\right)^6$$

By Pascal triangle, Binomial coefficients for the index 6 are 1, 6, 15, 20, 15, 6, 1

$$= 1.x^6 - 6x^5\left(\frac{1}{x}\right) + 15x^4\left(\frac{1}{x^2}\right) - 20x^3\left(\frac{1}{x^3}\right) + 15x^2\left(\frac{1}{x^4}\right) - 6x\left(\frac{1}{x^5}\right) + 1\left(\frac{1}{x^6}\right)$$

$$= x^6 - 6x^4 + 15x^2 - 20 + 15\left(\frac{1}{x^2}\right) - 6\left(\frac{1}{x^4}\right) + \frac{1}{x^6}$$

$$2^6 i^6 \sin^6\theta = \left(x^6 + \frac{1}{x^6}\right) - 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) - 20$$

$$-2^6\sin^6\theta = 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20$$

$$\sin^6\theta = -\frac{1}{64}\cdot 2\{\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\} \qquad \{\because i^6 = -1$$

$$2^6 = 64$$

$$\boxed{\sin^6\theta = -\frac{1}{32}\{\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\}}$$

Example 3.6 Prove that $64(\cos^8\theta + \sin^8\theta) = \cos 8\theta + 28\cos 4\theta + 35$

Proof

Let $x = \cos\theta + i\sin\theta \Rightarrow \frac{1}{x} = \cos\theta - i\sin\theta$

$$x + \frac{1}{x} = 2\cos\theta, \quad x - \frac{1}{x} = 2i\sin\theta$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta, \quad x^n - \frac{1}{x^n} = 2i\sin n\theta$$

we have,

$$2\cos\theta = x + \frac{1}{x}$$

Raising to the power 8

$$2^8\cos^8\theta = \left(x + \frac{1}{x}\right)^8$$

$$= x^8 + 8x^7\left(\frac{1}{x}\right) + 28x^6\left(\frac{1}{x^2}\right) + 56x^5\left(\frac{1}{x^3}\right) + 70x^4\left(\frac{1}{x^4}\right) + 56x^3\left(\frac{1}{x^5}\right) + 28x^2\left(\frac{1}{x^6}\right) + 8x\left(\frac{1}{x^7}\right) + \frac{1}{x^8}$$

$$= x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + 56\left(\frac{1}{x^2}\right) + 28\left(\frac{1}{x^4}\right) + 8\left(\frac{1}{x^6}\right) + \frac{1}{x^8}$$

$$= \left(x^8 + \frac{1}{x^8}\right) + 8\left(x^6 + \frac{1}{x^6}\right) + 28\left(x^4 + \frac{1}{x^4}\right) + 56\left(x^2 + \frac{1}{x^2}\right) + 70$$

$$2(2^7\cos^8\theta) = 2\cos 8\theta + 8(2\cos 6\theta) + 28(2\cos 4\theta) + 56(2\cos 2\theta) + 70$$

Dividing both Sides by 2,

$$\boxed{2^7\cos^8\theta = \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35} \rightarrow (1)$$

$$2i\sin\theta = x - \frac{1}{x}$$

$$(2i)^8\sin^8\theta = \left(x - \frac{1}{x}\right)^8$$

$$= x^8 - 8x^7\left(\frac{1}{x}\right) + 28x^6\left(\frac{1}{x^2}\right) - 56x^5\left(\frac{1}{x^3}\right) + 70x^4\left(\frac{1}{x^4}\right) - 56x^3\left(\frac{1}{x^5}\right) + 28x^2\left(\frac{1}{x^6}\right) - 8x\left(\frac{1}{x^7}\right) + \frac{1}{x^8}$$

$$= \left(x^8 + \frac{1}{x^8}\right) - 8\left(x^6 + \frac{1}{x^6}\right) + 28\left(x^4 + \frac{1}{x^4}\right) - 56\left(x^2 + \frac{1}{x^2}\right) + 70$$

$$= 2\cos 8\theta - 8(2\cos 6\theta) + 28(2\cos 4\theta) - 56(2\cos 2\theta) + 70$$

$$2^8 i^8 \sin^8\theta = 2\{\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35\}$$

$\boxed{i^8 = 1}$

Dividing both sides by 2

$$\boxed{2^7 \sin^8\theta = \cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35} \longrightarrow (2)$$

Adding (1) and (2) we get

$$2^7(\cos^8\theta + \sin^8\theta) = 2\cos 8\theta + 56\cos 4\theta + 70$$

Dividing both sides by 2

$\boxed{2^6 = 64}$

$$64(\cos^8\theta + \sin^8\theta) = \cos 8\theta + 28\cos 4\theta + 35$$

Hence the Proof.

Example 3.7

$$-256\sin^7\theta\cos^2\theta = \sin 9\theta - 5\sin 7\theta + 8\sin 5\theta - 14\sin\theta$$

Solution

Let $x = \cos\theta + i\sin\theta$ then $\frac{1}{x} = \cos\theta - i\sin\theta$

So that $x + \frac{1}{x} = 2\cos\theta$, $x - \frac{1}{x} = 2i\sin\theta$

$$\boxed{\left(x + \frac{1}{x}\right)^2 = 2^2\cos^2\theta}, \quad \left(x - \frac{1}{x}\right)^7 = (2i)^7 \sin^7\theta$$

$$= 2^7(-i)\sin^7\theta$$

$$\boxed{\left(x - \frac{1}{x}\right)^7 = -i\,2^7\sin^7\theta}$$

$$\left(x - \frac{1}{x}\right)^7\left(x + \frac{1}{x}\right)^2 = (-i\,2^7\sin^7\theta)(2^2\cos^2\theta)$$

$$\left(x - \frac{1}{x}\right)^7\left(x + \frac{1}{x}\right)^2 = -i\,2^9\sin^7\theta\cdot\cos^2\theta$$

$$\Rightarrow \boxed{-i\,2^9\sin^7\theta\cos^2\theta = \left(x - \frac{1}{x}\right)^7\left(x + \frac{1}{x}\right)^2} \longrightarrow (1)$$

///

$$\left(x-\frac{1}{x}\right)^7\cdot\left(x+\frac{1}{x}\right)^2 = \left(x^7-7x^5+21x^3-35x+\frac{35}{x}-\frac{21}{x^3}+\frac{7}{x^5}-\frac{1}{x^7}\right)\cdot\left(x^2+2+\frac{1}{x^2}\right)$$

$$= \cancel{x^9+2x^7}\; x^9-7x^7+21x^5-35x^3+35x-\frac{21}{x}+\frac{7}{x^3}-\frac{1}{x^5}$$

$$+2x^7-14x^5+42x^3-70x+\frac{70}{x}-\frac{42}{x^3}+\frac{14}{x^5}-\frac{2}{x^7}$$

$$+x^5-7x^3+21x-\frac{35}{x}+\frac{35}{x^3}-\frac{21}{x^5}+\frac{7}{x^7}-\frac{1}{x^9}$$

$$= \left(x^9-\frac{1}{x^9}\right)-5\left(x^7-\frac{1}{x^7}\right)+8\left(x^5-\frac{1}{x^5}\right)-14\left(x-\frac{1}{x}\right)$$

$$= 2i\sin 9\theta - 5(2i\sin 7\theta)+8(2i\sin 5\theta)-14(2i\sin\theta)$$

$$\boxed{\left(x-\frac{1}{x}\right)^7\left(x+\frac{1}{x}\right)^2 = 2i\left(\sin 9\theta-5\sin 7\theta+8\sin 5\theta-14\sin\theta\right)} \longrightarrow (2)$$

Substitute (2) in (1)

$$-i\,2^9\sin^7\theta\cos^2\theta = 2i\left(\sin 9\theta-5\sin 7\theta+8\sin 5\theta-14\sin\theta\right)$$

$$-2^8\sin^7\theta\cos^2\theta = \left(\sin 9\theta-5\sin 7\theta+8\sin 5\theta-14\sin\theta\right)$$

$$\boxed{-256\sin^7\theta\cos^2\theta = \sin 9\theta-5\sin 7\theta+8\sin 5\theta-14\sin\theta}$$

Hence the Proof.

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Expansion of $\cos n\theta$ and $\sin n\theta$ in ascending power of θ.

By Demoivre's theorem $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

By Binomial expansion,

$$(\cos\theta + i\sin\theta)^n = (\cos\theta)^n + \binom{n}{1}(\cos\theta)^{n-1}(i\sin\theta) + \binom{n}{2}(\cos\theta)^{n-2}(i\sin\theta)^2 + \binom{n}{3}(\cos\theta)^{n-3}(i\sin\theta)^3 + \binom{n}{4}(\cos\theta)^{n-4}(i\sin\theta)^4$$
$$+ \cdots + \binom{n}{n}(i\sin\theta)^n$$

$$\cos n\theta + i\sin n\theta = \cos^n\theta + i\binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{2}\cos^{n-2}\theta \cdot \sin^2\theta - i\binom{n}{3}\cos^{n-3}\theta\sin^3\theta$$
$$+ \binom{n}{4}\cos^{n-4}\theta\sin^4\theta + \cdots + (i)^n\sin^n\theta$$

Note:
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = -i$
$i^6 = -1$
$\vdots$

Equating the real and imaginary parts, we get

Real Part: $\cos n\theta = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta \sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta \cdots$

Imaginary Part: $\sin n\theta = \binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \cdots$

Example 3.8 Expand $\cos 7\theta$ and $\sin 7\theta$ in ascending power of θ

Solution

$$(\cos\theta + i\sin\theta)^7 = \cos^7\theta + \binom{7}{1}\cos^6\theta(i\sin\theta) + \binom{7}{2}\cos^5\theta(i\sin\theta)^2 + \binom{7}{3}\cos^4\theta(i\sin\theta)^3 + \binom{7}{4}\cos^3\theta(i\sin\theta)^4$$
$$+ \binom{7}{5}\cos^2\theta(i\sin\theta)^5 + \binom{7}{6}\cos\theta(i\sin\theta)^6 + \binom{7}{7}(i\sin\theta)^7$$

$$\cos 7\theta + i\sin 7\theta = \cos^7\theta + i7\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - i\,35\cos^4\theta\sin^3\theta + 35\cos^3\theta\sin^4\theta$$
$$+ i\,21\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - i\sin^7\theta$$

Expansion of $\cos n\theta$ and $\sin n\theta$ in ascending power of θ.

By Demoivre's theorem $\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

By Binomial expansion,

$$(\cos\theta + i\sin\theta)^n = (\cos\theta)^n + \binom{n}{1}(\cos\theta)^{n-1}(i\sin\theta) + \binom{n}{2}(\cos\theta)^{n-2}(i\sin\theta)^2 + \binom{n}{3}(\cos\theta)^{n-3}(i\sin\theta)^3 + \binom{n}{4}(\cos\theta)^{n-4}(i\sin\theta)^4$$
$$+ \cdots + \binom{n}{n}(i\sin\theta)^n$$

$$\cos n\theta + i\sin n\theta = \cos^n\theta + i\binom{n}{1}\cos^{n-1}\theta\,\sin\theta - \binom{n}{2}\cos^{n-2}\theta\cdot\sin^2\theta - i\binom{n}{3}\cos^{n-3}\theta\,\sin^3\theta$$
$$+ \binom{n}{4}\cos^{n-4}\theta\,\sin^4\theta + \cdots + (i)^n\sin^n\theta$$

Note:
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = -i$
$i^6 = -1$
$\vdots$

Equating the real and imaginary parts, we get

Real Part: $\cos n\theta = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\,\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\,\sin^4\theta \cdots$

Imaginary Part: $\sin n\theta = \binom{n}{1}\cos^{n-1}\theta\,\sin\theta - \binom{n}{3}\cos^{n-3}\theta\,\sin^3\theta + \cdots$

Example 3.8 Expand $\cos 7\theta$ and $\sin 7\theta$ in ascending power of θ

Solution

$$(\cos\theta + i\sin\theta)^7 = \cos^7\theta + \binom{7}{1}\cos^6\theta\,(i\sin\theta) + \binom{7}{2}\cos^5\theta\,(i\sin\theta)^2 + \binom{7}{3}\cos^4\theta\,(i\sin\theta)^3 + \binom{7}{4}\cos^3\theta\,(i\sin\theta)^4$$
$$+ \binom{7}{5}\cos^2\theta\,(i\sin\theta)^5 + \binom{7}{6}\cos\theta\,(i\sin\theta)^6 + \binom{7}{7}(i\sin\theta)^7$$

$$\cos 7\theta + i\sin 7\theta = \cos^7\theta + i7\cos^6\theta\,\sin\theta - 21\cos^5\theta\,\sin^2\theta - i\,35\cos^4\theta\,\sin^3\theta + 35\cos^3\theta\,\sin^4\theta$$
$$+ i\,21\cos^2\theta\,\sin^5\theta - 7\cos\theta\,\sin^6\theta - i\sin^7\theta$$

Equating the real and imaginary parts, we get

Real Part : ~~Cos 7θ =~~

$$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\,\sin^2\theta + 35\cos^3\theta\,\sin^4\theta - 7\cos\theta\,\sin^6\theta$$

$$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-\cos^2\theta)^2 - 7\cos\theta(1-\cos^2\theta)^3$$ {∵ Convert it interms of $\cos\theta$

$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta(1+\cos^4\theta - 2\cos^2\theta) - 7\cos\theta(1-3\cos^2\theta+3\cos^4\theta - \cos^6\theta)$$

$$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta + 35\cos^7\theta - 70\cos^5\theta - 7\cos\theta + 21\cos^3\theta - 21\cos^5\theta + 7\cos^7\theta$$

$$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

$$\boxed{\cos 7\theta = -7\cos\theta + 56\cos^3\theta - 112\cos^5\theta + 64\cos^7\theta}$$ (in ascending order

Imaginary Part :

$$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$$ (∵ Convert it, interms of $\sin\theta$

$$= 7(1-\sin^2\theta)^3\sin\theta - 35(1-\sin^2\theta)^2\sin^3\theta + 21(1-\sin^2\theta)\sin^5\theta - \sin^7\theta$$

$$= 7(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta)\sin\theta - 35(1-2\sin^2\theta+\sin^4\theta)\sin^3\theta + 21(1-\sin^2\theta)\sin^5\theta - \sin^7\theta$$

$$= 7\sin\theta - 21\sin^3\theta + 21\sin^5\theta - 7\sin^7\theta - 35\sin^3\theta + 70\sin^5\theta - 35\sin^7\theta + 21\sin^5\theta - 21\sin^7\theta - \sin^7\theta$$

$$\boxed{\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta}$$

Example 3.9 Prove that $2(1+\cos 8\theta) = (x^4 - 4x^2 + 2)^2$ where $x = 2\cos\theta$

Proof

$2(1+\cos 8\theta) = 2(2\cos^2 4\theta) = 4\cos^2 4\theta \longrightarrow (1)$ $\quad \{\because 1+\cos 2\theta = 2\cos^2\theta$

$(\cos\theta + i\sin\theta)^4 = \cos^4\theta + \binom{4}{1}\cos^3\theta\cdot(i\sin\theta) + \binom{4}{2}\cos^2\theta(i\sin\theta)^2 + \binom{4}{3}\cos\theta(i\sin\theta)^3 + \binom{4}{4}(i\sin\theta)^4$

$= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i^2\sin^2\theta) + 4\cos\theta(i^3\sin^3\theta) + i^4\sin^4\theta$

$\cos 4\theta + i\sin 4\theta = \cos^4\theta + i\,4\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - i\,4\cos\theta\sin^3\theta + \sin^4\theta$

Equating the Real Part,

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$$\boxed{\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1}$$

(1) $\Rightarrow$ $2(1+\cos 8\theta) = \cancel{2\cdot 2\cos}\ 4\cos^2 4\theta$

$= (2\cos 4\theta)^2$

$= (16\cos^4\theta - 16\cos^2\theta + 2)^2$

$= [(2\cos\theta)^4 - 4(2\cos\theta)^2 + 2]^2$ $\quad \{\because x = 2\cos\theta$

$$\boxed{2(1+\cos 8\theta) = (x^4 - 4x^2 + 2)^2}$$

Example 3.10 Prove that (i) $2^{n/2} \cos \frac{n\pi}{4} = 1 - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots$

(ii) $2^{n/2} \sin \frac{n\pi}{4} = \binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \cdots$

where n is a positive integer

Proof:

(i) We know that

$$\cos n\theta = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta \sin^2\theta + \binom{n}{4}\cos^{n-4}\theta \sin^4\theta - \cdots$$

Put $\theta = \frac{\pi}{4}$

$$\cos n\frac{\pi}{4} = \cos^n\frac{\pi}{4} - \binom{n}{2}\cos^{n-2}\frac{\pi}{4}\sin^2\frac{\pi}{4} + \binom{n}{4}\cos^{n-4}\frac{\pi}{4}\sin^4\frac{\pi}{4} - \cdots$$

$$= \left(\frac{1}{\sqrt{2}}\right)^n - \binom{n}{2}\left(\frac{1}{\sqrt{2}}\right)^{n-2}\cdot\left(\frac{1}{\sqrt{2}}\right)^2 + \binom{n}{4}\left(\frac{1}{\sqrt{2}}\right)^{n-4}\left(\frac{1}{\sqrt{2}}\right)^4 - \cdots$$

$$= \left(\frac{1}{\sqrt{2}}\right)^n - \binom{n}{2}\left(\frac{1}{\sqrt{2}}\right)^n + \binom{n}{4}\left(\frac{1}{\sqrt{2}}\right)^n - \cdots$$

$$\cos n\frac{\pi}{4} = \frac{1}{(\sqrt{2})^n}\left\{1 - \binom{n}{2} + \binom{n}{4} - \cdots\right\}$$

$$(\sqrt{2})^n \cos \frac{n\pi}{4} = 1 - \binom{n}{2} + \binom{n}{4} - \cdots$$

$$\boxed{(2)^{n/2} \cos \frac{n\pi}{4} = 1 - \binom{n}{2} + \binom{n}{4} - \cdots}$$

(ii) we know that

$$\sin n\theta = \binom{n}{1}\cos^{n-1}\theta \sin\theta - \binom{n}{3}\cos^{n-3}\theta \sin^3\theta + \binom{n}{5}\cos^{n-5}\theta \sin^5\theta - \cdots$$

Put $\theta = \frac{\pi}{4}$

$$\sin n\frac{\pi}{4} = \binom{n}{1}\cos^{n-1}\frac{\pi}{4}\sin\frac{\pi}{4} - \binom{n}{3}\cos^{n-3}\frac{\pi}{4}\sin^3\frac{\pi}{4} + \binom{n}{5}\cos^{n-5}\frac{\pi}{4}\sin^5\frac{\pi}{4} - \cdots$$

$$= \binom{n}{1}\left(\frac{1}{\sqrt{2}}\right)^{n-1}\left(\frac{1}{\sqrt{2}}\right) - \binom{n}{3}\left(\frac{1}{\sqrt{2}}\right)^{n-3}\left(\frac{1}{\sqrt{2}}\right)^3 + \binom{n}{5}\left(\frac{1}{\sqrt{2}}\right)^{n-5}\left(\frac{1}{\sqrt{2}}\right)^5 - \cdots$$

$$= \binom{n}{1}\frac{1}{(\sqrt{2})^n} - \binom{n}{3}\frac{1}{(\sqrt{2})^n} + \binom{n}{5}\frac{1}{(\sqrt{2})^n} - \cdots$$

$$\sin n\frac{\pi}{4} = \frac{1}{(\sqrt{2})^n}\left\{\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \cdots\right\}$$

$$\boxed{2^{n/2}\sin n\frac{\pi}{4} = \binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \cdots}$$